Calculation of Unsteady Transonic Flows Using the Integral Equation Method

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The basic integral equations for a harmonically oscillating airfoil in a transonic flow with shock waves are derived; the reduced frequency is assumed to be small. The problems associated with shock wave motion are treated using a strained coordinate system. The integral equation is linear and consists of both line integrals and surface integrals over the flowfield which are evaluated by quadrature. This leads to a set of linear algebraic equations that can be solved directly. The shock motion is obtained explicitly by enforcing the condition that the flow is continuous except at a shock wave. Results obtained for both lifting and nonlifting oscillatory flows agree satisfactorily with other accurate results.

Introduction

THE calculation of unsteady pressures on an airfoil at transonic speeds is a critical problem in aerodynamics because an accurate estimation of the pressure distribution is necessary for either an adequate flutter analysis or the prediction of other unsteady flight phenomena (such as gust response). In two-dimensional flow most of the essential features are represented by the nonlinear, unsteady, transonic, small-disturbance equation for the velocity potential. For a flutter analysis, it usually can be assumed that the airfoil is oscillating harmonically; the velocity potential then can be expanded as a Fourier series in time. This reduces the time-dependent governing equation to an infinite set of linear partial differential equations that are independent of time. The first of this set is the mean steady flow; it usually is assumed that the amplitude of the oscillation is sufficiently small that only the second equation in the set, the fundamental response, need be solved in order to obtain an adequate representation of the unsteady flow. Alternatively, the basic time-dependent, small-disturbance equation may be solved directly. A fundamental problem in transonic unsteady flow calculations in which shock waves are present is associated with the shock motion, since the movement of the shock will cause very large pressure fluctuations in the region traversed by the shock motion.

Presently available methods of solving the harmonically decomposed problem include the finite-difference calculation of Traci et al.[1] and Weatherhill et al.[2] In these methods no account is taken of the shock motion, the shock being assumed to remain essentially at its steady-state location. A serious drawback of these procedures is the occurrence of a severe numerical instability in the relaxation procedure used to solve the difference equations. This instability occurs at a critical Mach-number-dependent frequency beyond which the method diverges. A means of countering the difficulty is given by Hafez et al.[3]; however, the rate of convergence is slow. The only available direct solutions of the unsteady, transonic, small-disturbance equation are those using the ADI method of Ballhaus and Goorjian[4]; the shock motion appears naturally in the course of the computations. All of the preceding methods use shock-capturing algorithms. Shock-fitting variations of these procedures are given in Ref. 3 for the harmonic approach and by Yu et al.[5] for the direct ADI approach.

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An alternative method for the commonly used finite-difference solutions of steady transonic flow problems is the extended integral equation method developed by Nixon.[6] The integral equation method gives a correct discontinuous representation of shock waves without difficulty, and, using the concept of analytic continuation,[7] accurate treatment of the boundary conditions is obtained easily. Furthermore, it appears that the integral equation method[6] is computationally more rapid than the generally used finite-difference procedures. Although the method of Ref. 6 has been applied only to steady transonic flow problems, an earlier, approximate version[8] of the integral equation method is derived for shock-free oscillatory flows. Most of the fundamental theory regarding the derivation of the unsteady integral equation method is given in Ref. 8. In the present paper, the extension of the extended integral equation method[6] to include harmonically oscillating flows is developed. The difficulties associated with the shock motion are treated by the method of strained coordinates previously developed[9] for steady perturbation problems. In this technique, the coordinate system is distorted such that the shock always remains at the same location throughout the perturbation. The magnitude of the required straining, which is related directly to the magnitude of the shock movement, is found as part of the solution. The tangential boundary condition is treated by the concept of analytic continuation[7] or by standard thin airfoil theory, depending on the problem under consideration. Although the basic integral equation is derived for an arbitrary frequency, only examples at low frequency are computed because the assumption of low frequency leads to simplifications in computation.

Applications of the theory to the problem of a biconvex airfoil harmonically pulsating in thickness and to an NACA 0012 airfoil oscillating in pitch are given. The steady-state solutions for these computations are obtained using a shock-capturing finite-difference program. The results of these computations agree fairly well with results obtained by the direct ADI method of Ballhaus and Goorjian.[4]

The computing time of the present method, 15 cpu-s on a CDC 7600 for the pitching case, is about twice as fast as the method of Ref. 4, even though no real attempt has been made yet to optimize the program. Although these computing times are for the low-frequency approximation, preliminary studies indicate that the computing times for arbitrary frequencies would be of comparable magnitude.

Basic Equation

For a freestream velocity and Mach number of U_∞ and M_∞, respectively, and in a Cartesian coordinate system $(\bar{x},\bar{z})$ scaled with respect to the airfoil chord with $\bar{x}$ aligned with the freestream, the unsteady, transonic, small-disturbance equation is

$$(1-M_\infty^2)\bar{\phi}_{\bar{x}\bar{x}}+\bar{\phi}_{\bar{z}\bar{z}}-(2M_\infty^2 c/U_\infty)\bar{\phi}_{\bar{x}t}-(M_\infty^2 c^2/U_\infty^2)\bar{\phi}_{tt}$$

$$=(\gamma+1)M_\infty^q\bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{x}\bar{x}} \quad (1)$$

where $\bar{\phi}$ is the perturbation velocity potential, t is time, γ is the ratio of specific heats, and c is the airfoil chord; q is a transonic parameter for the small-disturbance equation. The boundary conditions are as follows:

1) The flow at the airfoil surface remains tangential to the moving airfoil surface.

2) The pressure is continuous off the wing, particularly across the wake.

3) The perturbation velocity potential vanishes at large distances upstream of the airfoil.

4) For subsonic trailing edge, the Kutta trailing-edge condition must be satisfied, namely, that the velocities at the trailing edge must remain finite.

For small-amplitude oscillations, the upper and lower surface of the airfoil are denoted by $\bar{z}=\bar{z}_u(\bar{x},t)$ and by $\bar{z}=\bar{z}_L(\bar{x},t)$, respectively. It now is assumed that the wing oscillates in simple harmonic motion with frequency ω and that the upper and lower surfaces of the wing at any time can be expressed as

$$\bar{z}_u(\bar{x},t)=\bar{z}_{0u}(\bar{x})+\bar{z}_{1u}(\bar{x})e^{i\omega t} \quad (2a)$$

$$\bar{z}_L(\bar{x},t)=\bar{z}_{0L}(\bar{x})+\bar{z}_{1L}(\bar{x})e^{i\omega t} \quad (2b)$$

where $\bar{z}=\bar{z}_{0u}(\bar{x})$ and $\bar{z}=\bar{z}_{0L}(\bar{x})$ represent the mean profile, and $\bar{z}_{1u}(\bar{x})$ and $\bar{z}_{1L}(\bar{x})$ represent the mode shape of the oscillation. The tangency boundary conditions are then

$$\frac{\bar{w}(\bar{x},\bar{z}_u,t)}{[1+\bar{u}(\bar{x},\bar{z}_u,t)]}=\frac{\partial\bar{z}_{0u}(\bar{x})}{\partial\bar{x}}+\left\{\frac{\partial\bar{z}_{1u}(\bar{x})}{\partial\bar{x}}+\frac{i\nu\bar{z}_{1u}(\bar{x})}{[1+\bar{u}(\bar{x},\bar{z}_u,t)]}\right\}e^{i\omega t} \quad (3a)$$

$$\frac{\bar{w}(\bar{x},\bar{z}_L,t)}{[1+\bar{u}(\bar{x},\bar{z}_L,t)]}=\frac{\partial\bar{z}_{0L}(\bar{x})}{\partial\bar{x}}+\left\{\frac{\partial\bar{z}_{1L}(\bar{x})}{\partial\bar{x}}+\frac{i\nu\bar{z}_{1L}(\bar{x})}{[1+\bar{u}(\bar{x},\bar{z}_L,t)]}\right\}e^{i\omega t} \quad (3b)$$

where

$$\bar{w}(\bar{x},\bar{z},t)=\frac{\partial\bar{\phi}(\bar{x},\bar{z},t)}{\partial\bar{z}} \quad (4a)$$

$$\bar{u}(\bar{x},\bar{z},t)=\frac{\partial\bar{\phi}(\bar{x},\bar{z},t)}{\partial\bar{x}} \quad (4b)$$

and $\nu=\omega c/U_\infty$, which is the reduced frequency.

As noted in the Introduction, one of the main difficulties associated with unsteady transonic flows when shock waves are present is to treat the shock motion correctly. In an allied investigation[9] for the perturbation of steady transonic flows, it is found that this problem is overcome easily by the use of a strained coordinate system in which the shock location is invariant. The main restriction is that no loss or generation of shock waves may occur during the perturbation. The theory is simplified if it is assumed that the shock wave is normal to the freestream, since in this case only the streamwise coordinate need be strained. For simplicity in presentation, it is assumed that there is only one shock wave on each surface of the airfoil. Following the basic ideas of Ref. 9, $\bar{x}$ then is expanded as a Fourier series in time; thus,

$$\bar{x}=x'+x_1(x',\pm 0)e^{i\omega t}+\ldots \quad (5)$$

where x' is the strained coordinate, and, from Ref. 9,

$$x_1(x',\pm 0)=\delta x_s(\pm 0)\hat{x}_1(x',\pm 0) \quad (6a)$$

$$\hat{x}_1(x',\pm 0)=\frac{x'(1-x')}{x_s'(\pm 0)[1-x_s'(\pm 0)]},\quad 0\le x'\le 1,$$

$$x_1(x',\pm 0)=0,\quad x'<0,\quad x'>1 \quad (6b)$$

where $x_1(x',\pm 0)$ is the straining function for the upper and lower surfaces, respectively; $x_s'(\pm 0)$ and $\delta x_s(\pm 0)$ denote the steady-state shock location and the magnitude of the shock oscillation in the upper and lower half-planes, respectively.

If $k=(\gamma+1)M_\infty^q$, $\beta=(1-M_\infty^2)^{1/2}$, and $z=\beta\bar{z}$, then $\bar{\phi}(\bar{x},\bar{z},t)$ is expanded in the Fourier series

$$\bar{\phi}(\bar{x},\bar{z},t)=(\beta^2/k)[\phi_0(x,z)+\phi_1(x',z)e^{i\omega t}$$

$$\times\exp(iM_\infty^2\Omega x')+\ldots] \quad (7)$$

where

$$\Omega=\nu/\beta^2 \quad (8)$$

Substitution of the series given by Eqs. (5) and (7) into Eq. (1) and equating coefficients of $e^{i\omega nt}(n=0,1)$ gives, for the first two equations of the infinite set,

$$\phi_{0_{x'x'}}+\phi_{0_{zz}}=\phi_{0_{x'}}\phi_{0_{x'x'}} \quad (9)$$

and

$$\phi_{1_{x'x'}}+\phi_{1_{zz}}+K^2\phi_1=g_{1_{x'}}+\exp(-iM_\infty^2\Omega x')$$

$$\times(f_{1_{x'}}+x_{1_{x'}}f_{2_{x'}}-2ix_{1_{x'}}M_\infty^2\Omega u_0) \quad (10)$$

where

$$u_0(x',z)=\frac{\partial\phi_0(x',z)}{\partial x'}$$

and

$$g_1(x',z)=\phi_{1_{x'}}^*(x',z)u_0(x',z)$$

$$+iM_\infty^2\Omega\int_{-\infty}^{x'}\phi_{1_\xi}^*(\xi,z)u_0(\xi,z)\mathrm{d}\xi \quad (11)$$

where

$$\phi_{1_{x'}}^*(x',z)=\phi_{1_{x'}}(x',z)+iM_\infty^2\Omega\phi_1(x',z) \quad (12)$$

and

$$K=M_\infty\Omega \quad (13)$$

Also,

$$f_1(x',z)=\{x_{1_{x'}}(x',\pm 0)[u_0(x',z)-u_0^2(x',z)]\} \quad (14a)$$

$$f_2(x',z)=\{u_0(x',z)-[u_0^2(x',z)/2]\} \quad (14b)$$

In order to use the integral equation formulation in its usual form, for which the boundary conditions must be satisfied on the chord line, the concept of an analytic continuation of the boundary conditions is used. Basically the exact boundary conditions on the airfoil surface are continued analytically to the chord line by means of a Taylor series; generally only the first two terms of the series are required. A full discussion of the ideas involved for steady flow is given in Ref. 7. For unsteady flow, the analytic continuation boundary conditions for Eqs. (9) and (10) are found to be

$$w_0(x',+0)=\bar{Z}_{0u_{x'}}(x')\left[1+\frac{\beta^2}{k}u_0(x',z_{0u})\right]$$

$$+z_{0u}(x')\frac{\partial}{\partial x'}\left[u_0(x',z_{0u})-\frac{u_0^2(x',z_{0u})}{2}\right] \quad (15a)$$

$$w_0(x',-0)=\bar{Z}_{0L_{x'}}(x')\left[1+\frac{\beta^2}{k}u_0(x',z_{0L})\right]$$

$$+z_{0L}(x')\frac{\partial}{\partial x'}\left[u_0(x',z_{0L})-\frac{u_0^2(x',z_{0L})}{2}\right] \quad (15b)$$

$$w_1(x',+0)=\left\{\bar{Z}_{1u_{x'}}(x')\left[1+\frac{\beta^2}{k}u_0(x',z_{0u})\right]+i\nu\bar{Z}_{1u}(x')\right\}$$

$$\times\exp(-iM_\infty^2\Omega x')+\frac{\beta^2}{k}\phi^*_{1_{x'}}(x',z_{0u})\bar{Z}_{0u_{x'}}(x')$$

$$-[z_{1u}(x')+x_1(x',+0)z_{1u_{x'}}(x')]\phi_{0_{zz}}(x',z_{0u})$$

$$\times\exp(-iM_\infty^2\Omega x')-z_{0u}(x')\phi_{1_{zz}}(x',z_{0u})$$

$$+x_1(x',+0)w_{0_{x'}}(x',+0)\exp(-iM_\infty^2\Omega x') \quad (16a)$$

$$w_1(x',-0)=\left\{\bar{Z}_{1L_{x'}}(x')\left[1+\frac{\beta^2}{k}u_0(x',z_{0L})\right]\right.$$

$$\left.+i\nu\bar{Z}_{1L}(x')\right\}\exp(-iM_\infty^2\Omega x')+\frac{\beta^2}{k}\phi^*_{1_{x'}}(x',z_{0L})$$

$$\times\bar{Z}_{0L_{x'}}(x')-[z_{1L}(x')+x_1(x',-0)z_{1L_{x'}}(x')]$$

$$\times\phi_{0_{zz}}(x',z_{0L})\exp(-iM_\infty^2\Omega x')-z_{0L}(x')\phi_{1_{zz}}(x',z_{0L})$$

$$+x_1(x',-0)w_{0_{x'}}(x',-0)\exp(-iM_\infty^2\Omega x') \quad (16b)$$

where

$$w_0(x',z)=\frac{\partial\phi_0(x',z)}{\partial z} \quad (17a)$$

$$w_1(x',z)=\frac{\partial\phi_1(x',z)}{\partial z} \quad (17b)$$

and

$$\bar{Z}_{0u}(x')=\frac{k}{\beta^3}\bar{z}_{0u}(\bar{x}),\quad \bar{Z}_{0L}(x')=\frac{k}{\beta^3}\bar{z}_{0L}(\bar{x}) \quad (18a)$$

$$\bar{Z}_{1u}(x')=\frac{k}{\beta^3}\bar{z}_{1u}(\bar{x}),\quad \bar{Z}_{1L}(x')=\frac{k}{\beta^3}\bar{z}_{1L}(\bar{x}) \quad (18b)$$

The terms $\phi_{0_{zz}}(x',z)$, $\phi_{1_{zz}}(x',z)$ are found in terms of the more easily computed x' derivatives by using Eqs. (9) and (10). The usual thin airfoil boundary conditions can be deduced from Eqs. (15) and (16) by neglecting the nonlinear terms.

The pressure coefficient $C_p(\bar{x},\bar{z},t)$ can be found from the unsteady Bernoulli equation; to the order of approximation of Eq. (1), then

$$C_p(\bar{x},\bar{z},t)=-2[\bar{\phi}_{\bar{x}}(\bar{x},\bar{z},t)+(c/U_\infty)\bar{\phi}_t(\bar{x},\bar{z},t)] \quad (19)$$

$C_p(\bar{x},\bar{z},t)$ may be expanded in a time-dependent series such as $\bar{\phi}(\bar{x},\bar{z},t)$; thus,

$$C_p(\bar{x},\bar{z},t)=\sum_{n=0}^{\infty}C_{pn}(x',z)e^{i\omega nt} \quad (20)$$

Noting that the straining is zero in the wake, then the unsteady pressure coefficient in the wake is given to a first approximation by the second term of the series Eq. (20); thus,

$$C_{p1}(x',z)=-2\frac{\beta^2}{k}[\phi_{1_{x'}}(x',z)+i\Omega\phi_1(x',z)]\exp(iM_\infty^2\Omega x')$$

The condition that there is no pressure jump across the wake therefore is given by

$$\Delta\phi_{1_{x'}}(x')+i\Omega\Delta\phi_1(x')=0,\quad x'\geq 1 \quad (21)$$

where the operator Δ for a function $f(x',z)$ is defined by

$$\Delta f(x')=f(x',+0)-f(x',-0) \quad (22)$$

Equation (21) can be solved to give the wake relation

$$\Delta\phi_1(x')=\Delta\phi_1(1)\exp[i\Omega(1-x')],\quad x'\geq 1 \quad (23)$$

The mean steady-state problem defined by Eqs. (9) and (15) is identical to the steady problem except for the introduction of the strained variable x'. It is assumed that this steady-state solution is known, and hence it will not be considered further.

The normal weak shock jump relation for Eq. (10) is found to be

$$[\phi_{1_{x'}}-g_1-\exp(-iM_\infty^2\Omega x')$$

$$(f_1+x_{1_{x'}}f_2-2ix_{1_{x'}}M_\infty^2\Omega\phi_0)]^+_-=0 \quad (24a)$$

$$[\phi_1]^+_-=0 \quad (24b)$$

The normal shock relations for Eq. (9) are

$$[u_0-(u_0^2/2)]^+_-=0 \quad (25a)$$

$$[\phi_0]^+_-=0 \quad (25b)$$

where $[\ ]^+_-$ denotes a jump across the shock wave. Substituting Eq. (25) into Eq. (24) and noting the definition of f_2 from Eq. (14b) leads to the simplified jump condition:

$$[\phi_{1_{x'}}-g_1-\exp(-iM_\infty^2\Omega x')f_1]^+_-=0 \quad (26a)$$

$$[\phi_1]^+_-=0 \quad (26b)$$

Equation (10) together with Eq. (26) can be written in integral form using Green's theorem. The details of the necessary analysis closely follow those given in Refs. 8 and 10. The integral equations for the derivatives of $\phi(x',z)$ can be found after some manipulation. Thus, using also Eq. (23),

$$\phi_{1_{x'}}(x',z)-g_1(x',z)=\int_0^1[\Psi_{x'}(Kr)]_{\zeta=0}\Delta\phi_{1_\zeta}(\xi)\,d\xi$$

$$+\int_0^1[\Psi_z(Kr)]_{\zeta=0}\Delta\phi_{1\xi}(\xi)\,d\xi$$

$$-i\Omega\Delta\phi_1(1)\int_1^\infty[\Psi_z(Kr)]_{\zeta=0}e^{i\Omega(1-\xi)}\,d\xi$$

$$-\int_s\int\Psi_{\xi x'}(Kr)g_1(\xi,\zeta)\,d\zeta\,d\xi+I_f(x',z) \quad (27)$$

where $\Psi(Kr)$ is the elementary solution of the equation

$$\Psi_{\xi\xi}+\Psi_{\zeta\zeta}+K^2\Psi=\delta(r) \quad (28)$$

where $\delta(r)$ is the delta function, and

$$r=[(x'-\xi)^2+(z-\zeta)^2]^{1/2} \quad (29)$$

The coordinates (ξ,ζ) correspond with the coordinates (x',z). Thus,

$$\Psi(Kr)=iH_0^{(2)}(Kr)/4 \quad (30)$$

where $H_0^{(2)}(Kr)$ is the Hankel function of zero order and of the second kind. $I_f(x',z)$ is given by

$$I_f(x',z)=[f_1(x',z)+x_{I_{x'}}(x',\pm 0)f_2(x',z)$$
$$-2iM_\infty^2\Omega x_{I_{x'}}(x',\pm 0)\phi_0(x',z)]\exp(-iM_\infty^2\Omega x')$$
$$-\int_s\int\{\Psi_{\xi x'}(Kr)[f_1(\xi,\zeta)+x_{I_\xi}(\xi,\pm 0)f_2(\xi,\zeta)$$
$$-2iM_\infty^2\Omega x_{I_\xi}(\xi,\pm 0)\phi_0(\xi,\zeta)]-iM_\infty^2\Omega\Psi_x(Kr)[f_1(\xi,\zeta)$$
$$+x_{I_\xi}(\xi,\pm 0)f_2(\xi,\zeta)-2iM_\infty^2\Omega x_{I_\xi}(\xi,\pm 0)\phi_0(\xi,\zeta)]$$
$$+[-2iM_\infty^2\Omega x_{I_{\xi\xi}}(\xi,\pm 0)\phi_0(\xi,\zeta)$$
$$+x_{I_{\xi\xi}}(\xi,\pm 0)f_2(\xi,\zeta)]\Psi_{x'}(Kr)\}\exp(-iM_\infty^2\Omega\xi)\mathrm{d}\zeta\,\mathrm{d}\xi \quad (31)$$

In Eqs. (27) and (31), the domain s of the integral is defined as

$$\int_s\int F\mathrm{d}\zeta\,\mathrm{d}\xi=\lim_{\substack{\delta\to 0\\ \epsilon\to 0}}\Big[\int_{-\infty}^{x'-\epsilon}\Big(\int_0^\infty F\mathrm{d}\zeta\Big)\mathrm{d}\xi+\int_{x'+\epsilon}^{x'_{s_u}-\delta}\Big(\int_0^\infty F\mathrm{d}\zeta\Big)\mathrm{d}\xi$$
$$+\int_{x'_{s_u}+\delta}^{\infty}\Big(\int_0^\infty F\mathrm{d}\zeta\Big)\mathrm{d}\xi+\int_{-\infty}^{x'-\epsilon}\Big(\int_{-\infty}^0 F\mathrm{d}\zeta\Big)\mathrm{d}\xi$$
$$+\int_{x'+\epsilon}^{x'_{s_L}-\delta}\Big(\int_{-\infty}^0 F\mathrm{d}\zeta\Big)\mathrm{d}\xi+\int_{x'_{s_L}+\delta}^{\infty}\Big(\int_{-\infty}^0 F\mathrm{d}\zeta\Big)\mathrm{d}\xi\Big] \quad (32)$$

where x'_{s_u}, x'_{s_L} are the locations of the shock waves on the upper and lower surfaces of the airfoil, respectively. As in the case of steady flow,[10] Eq. (27) degenerates to a symmetric (with respect to z) equation in the limit as $z\to\pm 0$. The necessary additional antisymmetric equation is obtained from the integral equation for $\phi_z(x',\pm 0)$, namely,

$$w_{I_T}(x')=-\int_0^1[\Psi_\xi(Kr)]_{\zeta=0}[\Delta\phi_{I_\xi}(\xi)-\Delta g_1(\xi)]\mathrm{d}\xi$$
$$+i\Omega\Delta\phi_1(1)\int_1^\infty[\Psi_\xi(Kr)]_{\zeta=0}e^{i\Omega(1-\xi)}\mathrm{d}\xi$$
$$-\int_s\int\Psi_{\xi z}(Kr)[g_1(\xi,\zeta)-\hat{g}_1(\xi,\zeta)]\mathrm{d}\zeta\,\mathrm{d}\xi+I_g(x') \quad (33)$$

where the operator Δ is defined by Eq. (22):

$$w_{I_T}(x')=\tfrac{1}{2}[\phi_{I_z}(x',+0)+\phi_{I_z}(x',-0)] \quad (34)$$

and

$$\hat{g}_1(\xi,\zeta)=\begin{cases}g_1(\xi,+0), & \zeta>0\\ g_1(\xi,-0), & \zeta<0\end{cases} \quad (35)$$

The term $I_g(x')$ is given by

$$I_g(x')=\lim_{z\to+0}\Big(-\int_s\int\{\Psi_{\xi z}(Kr)[f_1(\xi,\zeta)+x_{I_\xi}f_2(\xi,\zeta)$$
$$-2iM_\infty^2\Omega x_{I_\xi}(\xi,\pm 0)\phi_0(\xi,\zeta)]-iM_\infty^2\Omega\Psi_z(Kr)[f_1(\xi,\zeta)$$
$$+x_{I_\xi}(\xi,\pm 0)f_2(\xi,\zeta)-2iM_\infty^2\Omega x_{I_\xi}(\xi,\pm 0)\phi_0(\xi,\zeta)]$$
$$+\Psi_z(Kr)x_{I_{\xi\xi}}(\xi,\pm 0)[f_2(\xi,\zeta)-2iM_\infty^2\Omega\phi_0(\xi,\zeta)]\}$$
$$\times\exp(-iM_\infty^2\Omega\xi)\mathrm{d}\zeta\,\mathrm{d}\xi\Big) \quad (36)$$

Low-Frequency Approximation

The complete integral equation for unsteady transonic flow can be constructed from Eqs. (27) and (33). Such a construction does, however, require the inverse of the antisymmetric equation, Eq. (33); although this is possible, a much simpler analysis results if a low-frequency assumption is made, namely, that terms of order K^2 [including $K^2\ln(K)$] and higher can be neglected. In this case, the kernel $\Psi(Kr)$ reduces to the form

$$\Psi(Kr)=(1/2\pi)\ln(Kr)+\text{const}+O(K^2) \quad (37)$$

The constant vanishes during the necessary differentiations of the kernel. It is this definition of the kernel function which will be used in the remainder of the analysis. If the field integrals in the antisymmetric equation, Eq. (33), are assumed known and are taken over to the left-hand side, then Eq. (33) is in a form similar to that of the familiar incompressible equation, and the usual methods of inversion[11] can be applied. Thus, for low frequencies, the inversion of Eq. (33) gives

$$\Delta\phi_{I_{x'}}(x')+i\Omega\Delta\phi_1(x')-\Big[\Delta g_1(x')+i\Omega\int_{-\infty}^{x'}\Delta g_1(\xi)\mathrm{d}\xi\Big]$$
$$=\frac{2}{\pi}\Big[1-C\Big(\frac{\Omega}{2}\Big)\Big]\Big(\frac{1-x'}{x'}\Big)^{1/2}\Big[\int_0^1\Big(\frac{\xi}{1-\xi}\Big)^{1/2}\hat{w}(\xi)\mathrm{d}\xi$$
$$-\int_0^1\Delta g_1(\xi)\mathrm{d}\xi\Big]+\frac{2}{\pi}\int_0^1\Big[\Big(\frac{1-x'}{x'}\Big)^{1/2}\Big(\frac{\xi}{1-\xi}\Big)^{1/2}$$
$$\times\frac{1}{(x'-\xi)}-\frac{i\Omega}{2}\Lambda(x',\xi)\Big]\hat{w}(\xi)\mathrm{d}\xi \quad (38)$$

where

$$\hat{w}(x')=w_{I_T}(x')+\int_s\int[\Psi_{\xi_z}(Kr)]_{z=0}[g_1(\xi,\zeta)$$
$$-\hat{g}(\xi,\zeta)]\mathrm{d}\zeta\,\mathrm{d}\xi-I_g(x') \quad (39)$$

$C(\Omega/2)$ is Theodorsen's function, defined by

$$C\Big(\frac{\Omega}{2}\Big)=\frac{H_1^{(2)}(\Omega/2)}{H_1^{(2)}(\Omega/2)+iH_0^{(2)}(\Omega/2)} \quad (40)$$

and

$$\Lambda(x',\xi)=\tfrac{1}{2}\ln\left\{\frac{(x'+\xi-2x'\xi)+[\xi x'(2\xi-1)(2x'-1)]^{1/2}}{(x'+\xi-2x'\xi)-[\xi x(2\xi-1)(2x'-1)]^{1/2}}\right\} \quad (41)$$

If the velocity potential $\phi(x',z)$ and the shock shift $\delta x_s(\pm 0)$ are split into real and imaginary parts, that is,

$$\phi_1(x',z)=\phi_1^R(x',z)+i\phi_1^I(x',z) \quad (42a)$$

$$\delta x_s(\pm 0)=\delta x_s^R(\pm 0)+i\delta x_s^I(\pm 0) \quad (42b)$$

where a superscript R denotes the real part and a superscript I denotes the imaginary part, then the following comments are valid.

If the frequency is zero, the problem reduces to one of steady state, the solution of which is real. Hence the imaginary parts in Eq. (42), that is, $\phi_{I_{x'}}^I(x',z)$ and $\delta x_s^I(\pm 0)$, are frequency-dependent and reduce to zero as the frequency goes to zero. In the present low-frequency analysis, therefore, terms such as $\Omega\phi_{I_{x'}}^I(x',z)$ and $\Omega\delta x_s^I$ are assumed to be negligible. If this assumption is made, the low-frequency approximation to the unsteady transonic integral equations is, after much algebraic manipulation,

$$\phi^R_{1_{x'}}(x',z)[1-u_0(x',z)] = I^R_L(x',z) + I^R_s(x',z) + I^R_h(x',z,\delta x^R_s) \tag{43}$$

$$\phi^I_{1_{x'}}(x',z)[1-u_0(x',z)] = I^I_L(x',z) + I^I_s(x',z) + I^I_h(x',z,\delta x^I_s) \tag{44}$$

where, as before the superscripts R and I denote real and imaginary parts, respectively. The definitions of $I_L(x',z)$, $I_s(x',z)$, and $I_h(x',z,\delta x_s)$ are given in the Appendix, where it may be seen that Eq. (43) is independent of $\phi^I_1(x',z)$ and its derivatives and hence can be solved independently. The value of $\phi^R_1(x',z)$ and its derivatives then can be used in the computation of $\phi^I_1(x',z)$. It is seen that Eq. (43) will give an infinite value for $\phi^R_{1_{x'}}$ when u_0 is unity unless the right-hand side is zero at that point. This gives the necessary equations for $\delta x^R_s(\pm 0)$, namely,

$$I^R_h[x_0(\pm 0),z,\delta x^R_s(\pm 0)] = -\{I^R_L[x_0(\pm 0),z] + I^R_s[x_0(\pm 0),z]\} \tag{45}$$

where $x' = x_0(\pm 0)$ denotes the line where $u_0 = 1$ on the upper and lower half-planes, respectively. Application of the relation Eq. (45) at $x_0(+0)$ and $x_0(-0)$ gives the necessary equations for $\delta x^R_s(\pm 0)$. A similar relation also applies for $\delta x^I_s(\pm 0)$; thus,

$$I^I_h[x_0(\pm 0),z,\delta x^I_s(\pm 0)] = -\{I^I_L[x_0(\pm 0),z] + I^I_s[x_0(\pm 0),z]\} \tag{46}$$

If $\phi_{1_{x'}}(x',z)$ and $\delta x_s(\pm 0)$ are known, then the total value $\bar{\phi}_x(\bar{x},\bar{z},t)$ is found by the relation

$$\begin{aligned}\bar{\phi}_{\bar{x}}(\bar{x},\bar{z},t) = (\beta^2/k)\{&\phi_{0_{x'}}(x',z) \\ &\times[1-\delta x_s(\pm 0)x_{1_{x'}}(x',\pm 0)e^{i\omega t}] \\ &+e^{i\omega t}[\phi_{1_{x'}}(x',z)+iM^2_\infty\Omega\phi_1(x',z)]\exp(iM^2_\infty\Omega x')\}\end{aligned} \tag{47}$$

and

$$\bar{\phi}(\bar{x},\bar{z},t) = (\beta^2/k)[\phi_0(x',z) + e^{i\omega t}\exp(iM^2_\infty\Omega x')\phi_1(x',z)] \tag{48}$$

where

$$\bar{x} = x' + e^{i\omega t}x_1(x',\pm 0) \tag{49}$$

The real and imaginary components of $\bar{\phi}_{x'}(\bar{x},\bar{z},t)$ and $\bar{\phi}(\bar{x},\bar{z},t)$ can be found easily from Eqs. (47) and (48). The pressure coefficient then is given by Eq. (19). Equations (43), (45), and Eqs. (44), (46) are linear integral equations of $\phi^R_{1_{x'}}(x',z)$ and $\phi^I_{1_{x'}}(x',z)$, respectively, and are similar in form to the steady perturbation equations, the numerical solution of which is given in Ref. 12.

Briefly, the line and field integrals and reduced to finite sums of the unknowns at selected points in the flowfield. In the case of the field integral, the flowfield is divided into strips parallel to the x' axis, and the variation in the z direction of the flow variables in each strip is approximated in terms of values on the strip edges by linear interpolation. The field integrals then are reduced to line integrals by integrating with respect to ζ. These line integrals then are evaluated by standard means. Having reduced the integral equations to a set of linear algebraic equations by this means, the set of equations then is solved directly using Gaussian elimination. The regularity conditions, Eqs. (45) and (46), are incorporated directly into the governing set of equations. The details of this procedure are given in Ref. 12.

Results

The unsteady flow about two sinusoidally oscillating airfoils has been calculated using the preceding method: 1) the flow about a nonlifting 10% biconvex airfoil at $M_\infty = 0.808$ pulsating in thickness between 9% and 11% of chord with a frequency parameter of 0.1; and 2) a NACA 0012 airfoil at $M_\infty = 0.8$ oscillating in pitch about the midchord at a frequency parameter of 0.2; the amplitude of the oscillation is ½ deg, and the mean steady state is nonlifting. In both of these calculations, the exponent q in Eq. (1) is taken to be 2.0. The steady-state solution is calculated using a shock-capturing, finite-difference method. The boundary conditions used in the first example are the usual thin airfoil boundary conditions. In the second example, the analytic continuation boundary conditions of Eq. (16) are used.

In Fig. 1, the pressure distribution around the pulsating biconvex airfoil is compared to results calculated by the method of Ref. 4. The agreement is generally fairly satisfactory, although the pressures in the region of the shock when the shock location is forward are not so good. This disagreement probably is due to the effects of a fitted normal shock (compared to the shock capture of Ref. 4) and to the harmonic decomposition used in the present method. In Fig. 2 the shock motion is shown, and it can be seen that the comparison with the result of the method of Ref. 4 is adequate. The pressure distribution around the NACA 0012 airfoil is shown in Fig. 3 and the shock motion in Fig. 4. The comparison of the present results with the finite-difference results is satisfactory, and again the discrepancy in the pressure distribution near the shock is attributed mainly to the present use of a harmonic decomposition and normal shock-fitting.

The computing time for the pitching case, 15 cpu-s on the CDC 7600, is about twice as fast as the ADI[4] method. No

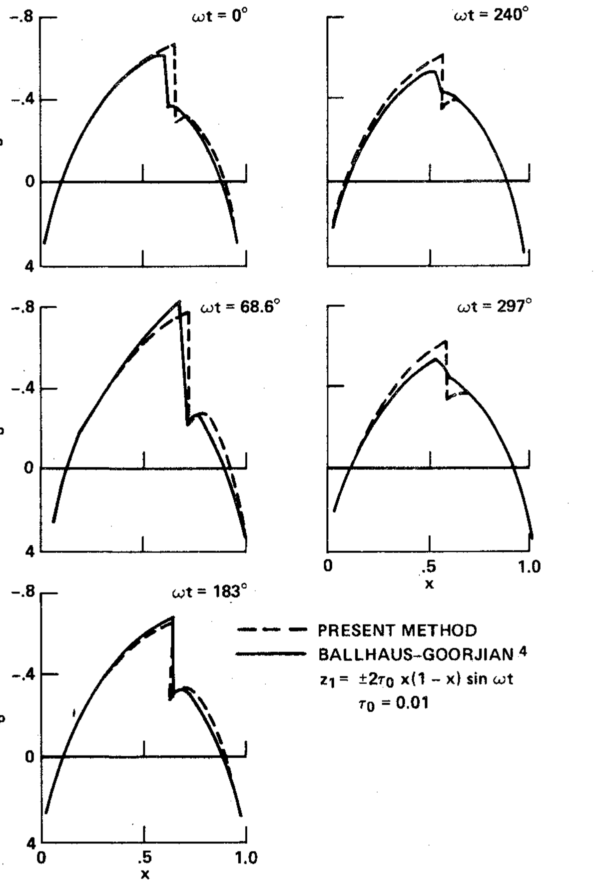

Fig. 1 Pressure distribution around a pulsating biconvex airfoil; $M_\infty = 0.808$, $\nu = 0.1$.

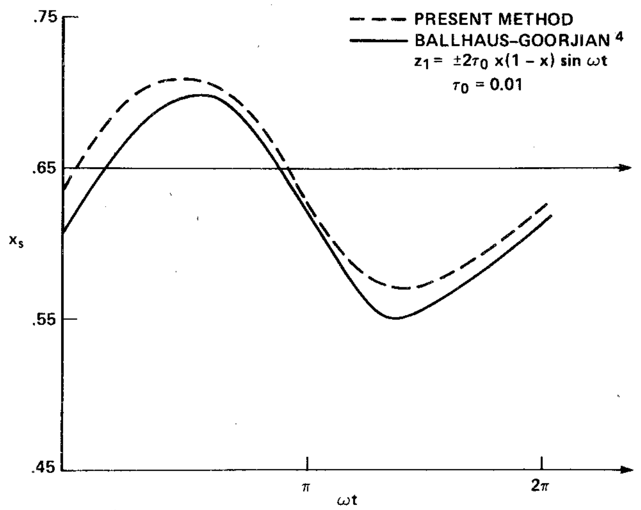

Fig. 2 Shock oscillation for a pulsating biconvex airfoil; $M_\infty = 0.808$, $\nu = 0.1$.

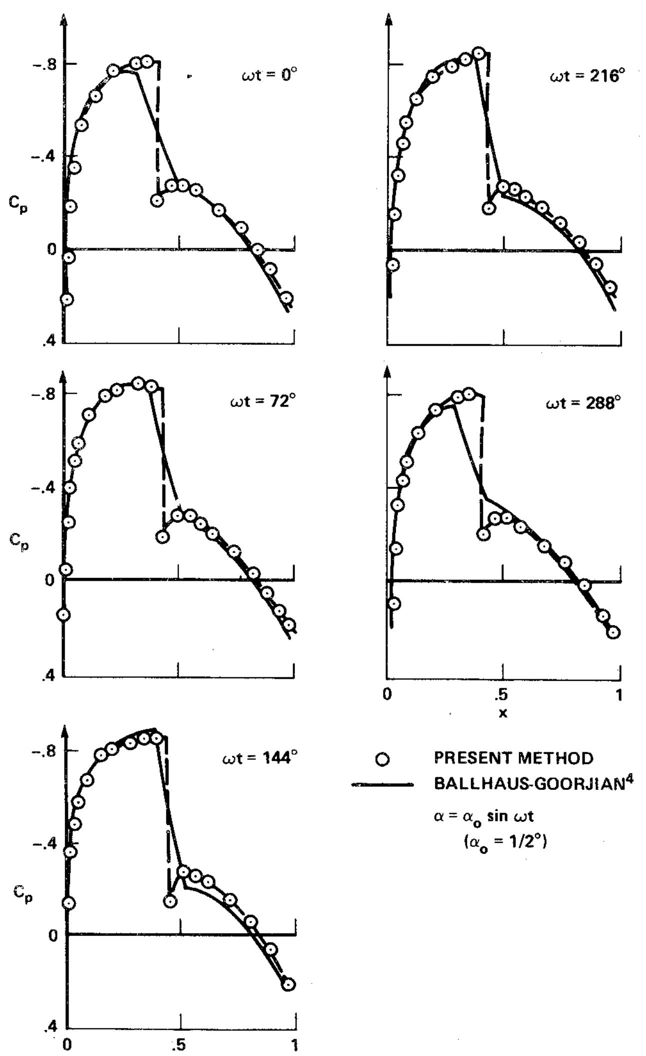

Fig. 3 Pressure distribution around the upper surface of a NACA 0012 airfoil pitching about midchord; $M_\infty = 0.8$, $\nu = 0.2$, $\alpha_0 = ½$ deg.

attempt has been made to optimize the program; such optimization probably would reduce further the computing time.

Concluding Remarks

The integral equation for unsteady transonic flow with shock waves is derived; the shock motion is treated by means of a strained coordinate system. A simplified form of the integral equation for low frequencies is derived and two examples computed, the results of which agree satisfactorily with other accurate calculations. Although the computations are performed only for the low-frequency equation, there is no fundamental barrier to a computation of the abritrary frequency case except perhaps the formidable amount of algebraic manipulation required. However, preliminary investigations of the arbitrary frequency aspects indicate that the computation of the problem would take about 2-3 times that of the present method.

The advantages of the present method are that it is computationally rapid, gives automatic shock fitting, and probably can be extended to higer frequencies without too much extra computational effort. The principal disadvantage seems to be the restriction that shock waves cannot be lost or generated during the motion; this restriction thus eliminates study of the important nonlinear shock motion effects of transonic flow. However, the restriction should not be a factor in flutter analyses that depend on infinitesimal motions.

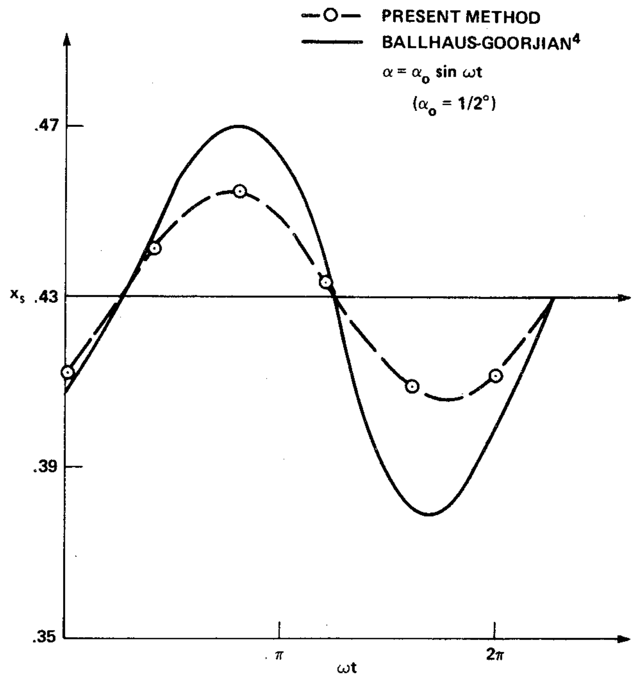

Fig. 4 Shock oscillation for a pitching NACA 0012 airfoil; $M_\infty = 0.8$, $\nu = 0.2$, $\alpha_0 = ½$ deg.

Appendix: Definition of Terms for the Low-Frequency Equation

In the definitions of the terms in the low-frequency equation, the following notation is used.

Functions

$$F_1(\xi,\zeta) = f_1(\xi,\zeta) + x_{1_\xi}(\xi, \pm 0) f_2(\xi,\zeta) \tag{A1}$$

where $f_1(\xi,\zeta)$ and $f_2(\xi,\zeta)$ are given by Eq. (14):

$$F_2(\xi,\zeta) = \phi_1(\xi,\zeta) u_0(\xi,\zeta) + \int_{-\infty}^{x'} \phi_{1_\xi}(\xi,\zeta) u_0(\xi,\zeta) \mathrm{d}\xi \tag{A2}$$

$$F_3(\xi,\zeta) = F_2(\xi,\zeta) - 2x_{1_\xi}(\xi, \pm 0)\phi_0(\xi,\zeta) - \xi F_1(\xi,\zeta) \tag{A3}$$

These functions also may be split into real and imaginary parts, e.g., $F_1 = F_1^R + iF_1^I$.

Integral Operators

For a function $f(\xi)$ or $f(\xi,\zeta)$, the following operators are defined:

$$L_1(x',\xi,f) = \frac{1}{\pi}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1\left(\frac{\xi}{1-\xi}\right)^{1/2} f(\xi)\,d\xi \tag{A4}$$

$$L_2(x',\xi,f) = \frac{1}{\pi}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1\left(\frac{\xi}{1-\xi}\right)^{1/2}\frac{f(\xi)}{(x'-\xi)}\,d\xi \tag{A5}$$

$$L_3(x',\xi,f) = \int_s\int(\Psi_{\xi_z})_{z=0}[f(\xi,\zeta)-\hat{f}(\xi,\zeta)]\,d\zeta\,d\xi \tag{A6}$$

where $\hat{f}(\xi,\zeta)$ is defined by

$$\hat{f}(\xi,\zeta) = \begin{cases} f(\xi,+0), & \zeta>0 \\ f(\xi,-0), & \zeta<0 \end{cases}$$

$$L_4(x',\xi,z,f) = -\int_s\int\Psi_{\xi_{x'}} f(\xi,\zeta)\,d\zeta\,d\xi \tag{A7}$$

$$L_5(x',\xi,f) = \int_s\int(\Psi_z)_{z=0} f(\xi,\zeta)\,d\zeta\,d\xi \tag{A8}$$

$$L_6(x',\xi,z,f) = -\int_s\int\Psi_{x'} f(\xi,\zeta)\,d\zeta\,d\xi \tag{A9}$$

where the integral over s is defined by Eq. (32), and the kernel Ψ is defined by Eq. (37).

Integrals

The following integrals now are defined:

$$I_1(x') = L_3(x',\xi,\phi_{1_\xi}u_0) \tag{A10}$$

$$I_2(x') = -L_3(x',\xi,F_1) - L_5(x',\xi x_{1_{\xi\xi}}f_2) \tag{A11}$$

$$I_3(x') = L_3(x',\xi,F_2+2x_{1_\xi}\phi_0+\xi F_1) + L_5(x',\xi,F_1+2x_{1_{\xi\xi}}\phi_0) \tag{A12}$$

As for the functions, these integrals also may be split into real and imaginary parts.

Algebraic Operators

The following algebraic operators on a function $f(\xi,\zeta)$ also are used:

$$\Delta_T f = \tfrac{1}{2}[f(\xi,+0)+f(\xi,-0)]$$

$$\Delta_c f = \tfrac{1}{2}[f(\xi,+0)-f(\xi,-0)]$$

The functions $I_L(x',z)$, $I_s(x',z)$, and $I_h(x',z,\delta x_s)$ now are defined as follows.

Real Part

For $z\neq 0$,

$$I_L^R(x',z) = \int_0^1(\Psi_z)_{\zeta=0}\Delta_c\phi_{1_\xi}^R\,d\xi + \int_0^1(\Psi_{x'})_{\zeta=0}\Delta_c\phi_{1_\zeta}^R\,d\xi \tag{A13}$$

$$I_s^R(x',z) = L_4(x',\xi,z,\phi_{1_\xi}^R u_0) \tag{A14}$$

$$I_h^R(x',z,\delta x_s) = L_4(x',\xi,z,F_1^R) + L_6(x',\xi,z,x_{1_{\xi\xi}}^R f_2) + F_1^R(x',z) \tag{A15}$$

and, for $z=\pm 0$,

$$I_L^R(x',\pm 0) = \int_0^1(\Psi_{x'})_{\substack{\zeta=0\\z=0}}\Delta_c\phi_{1_\zeta}^R\,d\xi \pm\left[\frac{\pi\Omega}{2}L_1(x',\xi,\Delta_T\phi_{1_\zeta}^R) + L_2(x',\xi,\Delta_T\phi_{1_\zeta}^R) - \frac{\pi\Omega}{2}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1\Delta_c(\phi_{1_\xi}^R u_0)\,d\xi\right] \tag{A16}$$

$$I_s^R(x',\pm 0) = L_4(x',\xi,0,\phi_{1_\xi}^R u_0) \pm [L_2(x',\xi,I_1^R) + (\pi\Omega/2)L_1(x',\xi,I_1^R)] \tag{A17}$$

$$I_h^R(x',\pm 0,\delta x_s) = L_4(x',\xi,0,F_1^R) + L_6(x',\xi,0,x_{1_{\xi\xi}}^R f_2) + F_1^R(x',\pm 0) \pm [L_2(x',\xi,I_2^R) + (\pi\Omega/2)L_1(x',\xi,I_2^R)] \tag{A18}$$

Imaginary Part

The imaginary part is given by the following relationships. For $z\neq 0$,

$$I_L^I(x',z) = \int_0^1(\Psi_z)_{\zeta=0}\Delta_c\phi_{1_\xi}^I\,d\xi + \int_0^1(\Psi_{x'})_{\zeta=0}\Delta_c\phi_{1_\zeta}^I\,d\xi - \Omega\Delta_c\phi_1^R(1)\int_1^\infty(\Psi_z)_{\zeta=0}\,d\xi + M_\infty^2\Omega[F_3^R(x',z) + L_4(x',\xi,z,F_3^R) - L_6(x',\xi,z,F_1^R+2x_{1_{\xi\xi}}\phi_0+\xi f_2)] \tag{A19}$$

$$I_s^I(x',z) = L_4(x',\xi,z,\phi_{1_\xi}^I u_0) \tag{A20}$$

$$I_h^I(x',z,\delta x_s) = L_4(x',\xi,z,F_1^I) + L_6(x',\xi,z,x_{1_{\xi\xi}}^I f_2) + F_1^I(x',z) \tag{A21}$$

and, for $z=\pm 0$,

$$\begin{aligned} I_L^I(x',\pm 0) = {} & \int_0^1(\Psi_{x'})_{\zeta=0}\Delta_c\phi_{1_\zeta}^I\,d\xi + M_\infty^2\Omega[F_3^R(x',\pm 0) - L_4(x',\xi,0,F_3^R) - L_6(x',\xi,0,F_1^R+2x_{1_{\xi\xi}}^R\phi_0)] \\ & \pm\{L_2(x',\xi,\Delta_T\phi_{1_\zeta}^I + M_\infty^2\Omega I_3^R) - \frac{\Omega}{2}\left[\ln\left(\frac{\Omega}{2}\right)+\gamma\right] \\ & \times[L_1(x',\xi,\Delta_T\phi_{1_\zeta}^R + I_1^R + I_2^R) - \left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1\Delta_c(\phi_{1_\xi}^R u_0)\,d\xi] \\ & - M_\infty^2\Omega I_h^R(x',\pm 0,\delta x_s^R) - \frac{\Omega}{2}\int_0^1\Lambda(x',\xi)(\Delta_T\phi_{1_\zeta}^R + I_1^R + I_2^R)\,d\xi \\ & + \Omega\delta L_2[x',\xi,\Delta_c\phi_1^R(1)] - \Omega\Delta_c\phi_1^R\} \\ & + \Omega\int_{-\infty}^{x'}\phi_{1_\xi}^R(\xi,\pm 0)u_0(\xi,\pm 0)\,d\xi \end{aligned} \tag{A22}$$

where

$$\delta = \ln\left(\frac{M_\infty}{2}\right) + \beta\ln\left|\frac{1+\beta}{M_\infty}\right|$$

In the preceding definition, γ is Euler's constant ($\gamma=0.577216$) and arises in the low-frequency expansion of Theodorsen's function $C(\Omega/2)$ of Eq. (40), and $\Lambda(x',\xi)$ is given by Eq. (41):

$$I_s^I(x',\pm 0) = L_4(x',\xi,0,\phi_{1_\xi}^I u_0) \pm [L_2(x',\xi,I_1^I)] \tag{A23}$$

$$I_h^I(x',\pm 0,\delta x_s) = L_4(x',\xi,0,F_1^I) + L_6(x',\xi,0,x_{1_{\xi\xi}}^I f_2) + F_1^I(x',\pm 0) \pm L_2(x',\xi,I_2^I) \tag{A24}$$

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